Pergamon

Int. J. Solids Structures Vol. 34, No. 5, pp. 545–568, 1997

0020–7683/96 $15.00 + .00

PII: S0020–7683(96)00034–0

SIZE EFFECTS IN THE DISTRIBUTION FOR STRENGTH OF BRITTLE MATRIX FIBROUS COMPOSITES

S. L. PHOENIX†, M. IBNABDELJALIL‡ and C.-Y. HUI†

† Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, U.S.A.

‡ Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.

(*Received 8 November 1995; in revised form 19 February 1996*)

Abstract — This paper addresses the effects of size (both length and width) on the probability distribution for strength of a composite consisting of brittle fibers aligned in a brittle matrix. The failure process involves quasi-periodic matrix cracking, frictional sliding of the fibers in fiber break zones, and fiber bridging of matrix cracks in a global load-sharing framework. The fiber strength follows the usual Poisson Weibull model of random flaws along the length. We first consider a composite cross-section, and develop the probability distribution for its strength in terms of certain characteristic fiber strength and length scales and the number of fibers in the cross-section. This strength distribution turns out to be a Gaussian distribution. We also calculate refined estimates of its mean and standard deviation, taking advantage of some new results based on an exact closed form solution for fragmentation of fibers in a single filament composite. We then consider the strength of a composite having a length orders of magnitude greater than the characteristic fiber length. We develop predictions for the scaling of the strength vs composite length based on certain results from the statistical theory of extremes in Gaussian processes. For this we develop an estimate of the covariance between the strengths of two nearby cross-sections of the composite. We also develop results based on a weakest-link analysis in terms of composite links of a certain length somewhat shorter than the characteristic fiber length. We then favorably compare our analytical results to numerical results from a Monte Carlo simulation of the composite failure process. This Monte Carlo model is free of various assumptions made in the analysis. The comparison suggests that predictions of a composite strength are possible for composite lengths orders of magnitude beyond what Monte Carlo simulation programs can currently handle. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

In recent years considerable attention has been paid to understanding the strength of brittle-matrix fibrous composites, that is, composites consisting of a glass or ceramic matrix reinforced by ceramic fibers in parallel. Addition of such fibers to a brittle monolithic matrix drastically changes the failure process away from one of extreme flaw sensitivity and variability, modeled reasonably well by Weibull-weakest volume statistics, to one of quasi-ductility, where the micromechanics of the failure process involves complex fiber and matrix load interactions and requires more sophisticated statistical modelling. Curtin (1991) describes the basic micromechanical assumptions of the model and introduces some key statistical ideas. As a foundation to the present work, the statistical ideas are developed variously by Phoenix and Raj (1992), Phoenix (1993), Curtin (1993), and Ibnabdeljalil and Phoenix (1995). A time dependent version, where fibers undergo creep rupture, was developed by Ibnabdeljalil and Phoenix (1995).

Briefly, the basic assumptions are: (i) quasi-periodic cracking of the matrix occurs perpendicular to the fibers and reaches saturation at stresses far less than the ultimate composite failure stress so that the matrix supports negligible tensile stress but transmits shear stresses laterally among fibers; (ii) a relatively low interfacial shear strength at the fiber-matrix interface exists (i.e. a constant interfacial shear strength) so that fibers slide frictionally within the matrix both within the matrix crack zones and near fiber breaks; (iii) fiber bridging of the matrix cracks occurs with redistribution of the original matrix stress and the stresses of broken fibers 'globally' or uniformly onto surviving fibers; and

(iv) as the load on the composite is increased, strain localization eventually develops around some cross-sectional plane that is statistically weaker than all the others, and the composite pulls apart as the fibers all break near that plane and pull out. The fiber strength is assumed to follow Poisson Weibull behavior having the usual weakest-link features. Also initial (before load application) Poisson random breaks can exist along the fiber with rate λ per unit length as studied by Phoenix and Ibnabdeljalil (1995), though in this paper we will focus on the case $\lambda = 0$.

Statistical analysis for the model builds on first considering the probability distribution for the strength of a composite segment of a certain characteristic fiber length δ_c and having n parallel fibers. The strength σ is normalized by a certain characteristic fiber stress σ_c. More specifically, the focus is on the normalized stress carrying capability of the segment's central cross-sectional plane and determining its probability distribution. The strength of this plane and the composite of length δ_c are taken as approximately equal as n becomes large. Generalizing the statistical ideas of equal load-sharing among n non-failed fibers as pioneered by Daniels (1945), to 'global' load-sharing where the slip zones next to a fiber break carry some load, Phoenix and Raj (1992) argued for an approximate normal (or Gaussian) distribution for the strength of this plane. They also discussed various approximations to the asymptotic normalized mean strength $\mu^*(n \to \infty)$, and considered small positive corrections to the asymptotic normalized mean Δ_n^* (decaying as $n^{-2/3}$) so that $\mu_n^* = \mu^* + \Delta_n^*$ is a more accurate representation of the mean strength for finite n. They also discussed approximations to the asymptotic normalized standard deviation γ_n^* (decaying as $n^{-1/2}$). All of these results were for the case $\lambda = 0$. The approximations all tend to agree and work well for a large Weibull shape parameter ρ for fiber strength, but as Ibnabdeljalil and Phoenix (1995) have demonstrated using a Monte Carlo simulation model of the failure process, they diverge and lose accuracy for $\rho < 4$, and the errors typically become large for $\rho < 2$. The major reason is that exclusion (shielded) zones around fiber breaks, where no further breakage can occur, are not properly accounted for in the approximations. Nevertheless, the strong tendency to a normal distribution for the strength of the composite segment of length δ_c was demonstrated. Some of these weaknesses in the various approximations can also be seen in the earlier Monte Carlo calculations of Curtin (1993). Ibnabdeljalil and Phoenix (1995) extended many of these approximations to the situation including initial discontinuities along the fiber ($\lambda > 0$), and again, inaccuracy arose as λ was increased such that the mean spacing of the initial breaks decreased to the order of δ_c.

To model the strength distribution for a composite of length L, orders of magnitude longer than δ_c, one idea has been to consider the composite to consist of a long chain of $m = L/\delta_c$ segments (global load-sharing bundles), each of length δ_c (Phoenix and Raj, 1992). The composite then fails when the weakest segment fails, and its strength distribution $H_{m,n}(\sigma/\sigma_c)$ is that of the weakest segment. The normalized strengths of the individual segments are treated as though they are independent and identically distributed (i.i.d.) random variables following a normal distribution with mean μ_n^* and standard deviation γ_n^*; that is, the strength distribution of an individual segment is $\Phi([(\sigma/\sigma_c)-\mu_n^*]/\gamma_n^*)$ where Φ is the standard normal distribution. The independence assumption seems reasonable since segments more than δ_c apart would appear to have negligible statistical interaction. Thus we would have $H_{m,n}(\sigma/\sigma_c) \approx 1-[1-\Phi([(\sigma/\sigma_c)-\mu_n^*]/\gamma_n^*)]^m$ based on weakest-link analysis. Taking advantage of well-known results in extreme value theory for a long sequence of i.i.d. normal random variables, Phoenix and Raj (1992) gave the asymptotic ($m \to \infty$) double exponential approximation for the strength distribution of the full composite $H_{m,n}(\sigma/\sigma_c)$ of length L. They also modified this result to yield a Weibull approximation for composite strength which is asymptotically equivalent ($n \to \infty$) and perhaps more useful. An early reference on the chain-of-bundles approach in materials failure is Gücer and Gurland (1962). Smith and Phoenix (1981) give some rigorous results for the strength of chains of statistically independent bundles of the Daniels type in that they quote precise conditions for proper convergence to the double exponential distribution in terms of increasing m and n. The intent was to get some idea of the accuracy of the extreme value approximation to $H_{m,n}(\sigma/\sigma_c)$, especially when m is large and n is small. No similar rigor, however, exists for the present model under global load-sharing.

While the above methodology has some foundation, it is based on various approximations, and questions naturally arise as to its accuracy. First, in the case of the simpler chain of equal load-sharing (Daniels) bundles, Phoenix and Raj (1992) give numerical results to suggest that the double exponential approximation to $H_{m,n}(\sigma/\sigma_c)$ is actually quite inaccurate for moderate bundle size n and large m especially in the lower tail (probability of failure of, say, 10^{-6}). Yet this, unfortunately, is where one would want to make predictions regarding high reliability of the composite. The Weibull variant of this approximation fares only slightly better. Fortunately both approximations appear to be conservative. Part of the difficulty is that the distribution for bundle strength is only *approximately* normally distributed, but it seems also that the double exponential distribution converges rather slowly as an asymptotic form (as m and $n \to \infty$), having an unrealistic shape in the lower tail. Second, it has been assumed that the effective bundle length (or length over which the strength of a given cross-sectional plane reasonably applies) is δ_c and that neighboring bundles along the chain are truly independent statistically. Thus far, there has been no good way to test these assumptions. Actually, for cross-sectional planes taken continuously along the composite, one might expect the strength at these planes to vary smoothly along the composite as approximately a Gaussian process, and that there should be some characteristic correlation function for strength in terms of distance between planes. Despite these unanswered questions, Curtin (1993) has performed some limited Monte Carlo calculations that suggest that the above methodology is reasonably accurate.

In what follows we will address the above issues, developing improvements to the various approximations that we will test against results from extensive Monte Carlo simulations. The goal is to determine improved forms that perform exceptionally well even in the lower tail (smaller σ/σ_c) of the cumulative distribution function $H_{m,n}(\sigma/\sigma_c)$ for long composites $L = m\delta_c$. First, we will calculate refined approximations to the asymptotic normalized mean μ^* and the standard deviation γ_n^* of the strength at a cross-section, taking advantage of some new results based on an exact closed form solution for fragmentation of fibers in a single filament composite, as obtained by Hui *et al.* (1995). These approximations will work well for virtually all $\rho > 0$. We will focus on the case $\lambda = 0$, though the results can easily be extended to the case $\lambda > 0$ using results in Hui *et al.* (1995). We will also give an estimate of Δ_n^*, the correction to the asymptotic mean μ^* to yield μ_n^*. Second, we will estimate the covariance function, later called $\Gamma_\delta(s^*)/n$, for the strengths of two cross-sectional planes separated by a small normalized distance δ and where s^* is a normalized fiber strain at cross-sectional collapse. With this covariance function in hand, we will be able to use results given in Leadbetter *et al.* (1983) for minima in Gaussian processes to determine the revised scale and location parameters for the double exponential approximation to the composite strength distribution $H_{m,n}(\sigma/\sigma_c)$ given in terms of m, n, μ^*, Δ_n^*, γ_n^* and $\Gamma_\delta(s^*)/n$.

We will also consider an alternative approximation for $H_{m,n}(\sigma/\sigma_c)$ based on the weakest-link form $H_{m,n}(\sigma/\sigma_c) \approx 1-[1-\Phi^*([(\sigma/\sigma_c)-\mu_n^*]/\gamma_n^*)]^{m'}$ where $m' = m/\beta$ and where the link length is taken as $\beta\delta_c$ for some constant $0 < \beta < 1$, which we try to estimate. In this weakest-link formula, we use an explicit and accurate approximation Φ^* to the lower tail of the Gaussian distribution representing the strength of a link as given in Cramér (1946). Its parameters are based on the location and scale parameters just mentioned for the double exponential approximation. We will finally compare these approximations favorably to numerical results from a Monte Carlo simulation model of the failure process, a model that is actually free of many of the assumptions made in deriving the approximations. We will also compare the weakest link and double exponential forms in terms of their accuracy in modelling the Monte Carlo distributions for strength at various composite lengths.

In Section 2 we introduce some basics of the model including the key scalings and normalizations for strength and length. In Section 3 we determine improved approximations for both the asymptotic mean μ^* (including a new quadratic approximation valid for very small $\rho > 0$) and the standard deviation γ_n^*, and determine an estimate for the covariance function $\Gamma_\delta(s^*)/n$. In Section 4 we develop the improved approximations for the distribution function for composite strength, $H_{m,n}(\sigma/\sigma_c)$. In Section 5 we compare these approximations to numerical results from Monte Carlo simulation. We close in Section 6 with some conclusions. The Appendix contains some analytical details of determining a Taylor series approximation for the asymptotic mean μ^* for very small $\rho > 0$.

2. BASIC MODEL AND SCALINGS FOR STRENGTH AND LENGTH

In this section we outline some of the basics of the model including key parameters, scalings and normalizations. For more details consult earlier works (Curtin, 1991; Phoenix and Raj, 1992; Curtin, 1993; Ibnabdeljalil and Phoenix, 1995).

We assume, that prior to any loading, flaws occur along the fiber according to a compound Poisson process in distance where the average number of flaws per unit length with strength less than σ is given by

$$\Lambda(\sigma) = (1/l_0)(\sigma/\sigma_0)^{\rho}. \tag{1}$$

This leads to the usual Poisson–Weibull model for fiber strength where $\sigma_0 > 0$ is the Weibull scale parameter relative to a test length l_0 of a continuous length in simple tension, and the exponent $\rho > 0$ is the usual Weibull shape parameter or modulus.

Around a fiber break or discontinuity at which the fiber stress is zero, the fiber length required to linearly build up to the far field fiber stress σ (basically the fiber Young's modulus times composite strain) is the slip length

$$l_f(\sigma) = r_f\sigma/(2\tau_s) \tag{2}$$

where r_f is the fiber radius and τ_s is the interfacial shear stress due to friction. Note that within this length, the fiber stress cannot be increased further so that no more breaks can occur; that is, the fiber is 'shielded'. In the failure process for the fibers we retain the normalizing scales for strength and length used in previous works, namely

$$\sigma_c = \sigma_0\{\tau_s l_0/(\sigma_0 r_f)\}^{1/(\rho+1)} \tag{3}$$

and

$$\delta_c = l_0\{\sigma_0 r_f/(l_0\tau_s)\}^{\rho/(\rho+1)}. \tag{4}$$

A simple interpretation of these scalings is that at the fiber stress level σ_c the mean number of Weibull flaws over the length δ_c, which have strengths less than or equal to σ_c, is exactly one. Also at the stress level σ_c, the length δ_c is exactly double the slip length $l_f(\sigma_c)$, that is, it is the total length around a fiber break where the stress is reduced or shielded.

These scales allow us to normalize all lengths by δ_c and all stresses by σ_c. Thus, we define a normalized stress s by

$$s = \sigma/\sigma_c, \tag{5}$$

and distance is always actual distance divided by δ_c. Then the mean number of flaws per unit dimensionless length at the dimensionless stress s is

$$\bar{\Lambda}(s) = s^{\rho}. \tag{6}$$

Furthermore, with these normalizations the slip zone on each side of a fiber break is $s/2$ when the normalized fiber stress is s.

With this review of some basics, we turn in the next section to the work of Hui *et al.* (1995) for some exact results on the statistics of fragmentation of fibers along a single filament composite. This is because in a composite with a large number of fibers ($n \to \infty$) we can think of each fiber as contained in an effective medium of fibers and matrix acting like a 'matrix' in the single filament composite. This idea will allow us to calculate some statistics for the effective load carrying of the fibers in the composite as the far field fiber stress s (or composite strain) is increased. As the composite strain is increased, the hazard rate for new breaks appearing with increasing s is

$$h(s) = \mathrm{d}\bar{\Lambda}(s)/\mathrm{d}s = \rho s^{\rho-1}, \quad s > 0. \tag{7}$$

However, the main complication in the analysis is that new breaks cannot occur in slip or shielded zones around old breaks. Also the stress carried by a fiber in a shielded zone is reduced depending on the distance to the nearest break, and shielded zones may overlap.

3. STATISTICS FOR COMPOSITE STRENGTH AT A CROSS-SECTION

Following the theme in earlier work (e.g. Phoenix and Raj, 1992; Curtin, 1993) the basic idea in analyzing the strength of the composite is to consider an applied far field fiber stress s, and consider the statistics of the load carried on a cross-sectional plane in terms of nearby breaks, that is, breaks close enough to the plane so that the load in the associated fibers would be reduced at that plane. The first quantity of interest is the asymptotic mean normalized stress function $\mu(s)$ for a large composite ($n \to \infty$) given $s > 0$. (Note that we ignore the tensile load carried by the matrix, which is typically negligible, and focus only on that in the fibers. Also the true 'effective' composite stress is really the stress we will calculate times the fiber volume fraction, f. This is to be understood in all our calculations.) Essentially, the main step is to determine $g(s, y)$, $0 \leqslant y < \infty$, the probability density function for the absolute distance from the plane to the nearest break along a given fiber. Then $\mu(s)$ is seen to be

$$\mu(s) = s\int_{s/2}^{\infty} g(s, y)\,\mathrm{d}y + \int_{0}^{s/2} 2yg(s, y)\,\mathrm{d}y, \tag{8}$$

where the first contribution corresponds to fibers that are not slipping (recall that the slip length is $s/2$) and the second contribution is the average stress of slipping fibers (a fiber at normalized distance y from a break is carrying normalized stress $2y$). In earlier works only approximations to $\mu(s)$ were calculated because $g(s, y)$ could only be approximated.

To determine $\mu(s)$ exactly, we turn to the analysis in Hui *et al.* (1995) and consider the quantity $p(s, x)$, which is the density function for the number per unit normalized length of inter-break spacings or fragments of length x under far field stress s. That is, in a fiber of length L, the number of fragments with lengths between x_1 and x_2 is, asymptotically,

$$L\int_{x_1}^{x_2} p(s, x)\,\mathrm{d}x \tag{9}$$

as L grows large, and the normalizing condition is

$$\int_{0}^{\infty} xp(s, x)\,\mathrm{d}x = 1. \tag{10}$$

Hui *et al.* (1995) give an exact, closed form solution to $p(s, x)$ (their eqn (21) in general and their (36) with (35) for the case at hand). It is easy to see that $xp(s, x)\,\mathrm{d}x$ is the probability that an arbitrary fragment cut by the cross-sectional plane of interest has length between x and $x+\mathrm{d}x$, and that the distance to its nearest end is uniformly distributed over 0 to $x/2$, so that

$$g(s, y) = \int_{2y}^{\infty} (2/x) x p(s, x) \, dx$$

$$= 2 \int_{2y}^{\infty} p(s, x) \, dx, \quad 0 \leqslant y < \infty. \tag{11}$$

Thus, $\mu(s)$ can be calculated by combining (8) and (11) and the result for $p(s, x)$ given in Hui *et al.* (1995) as just mentioned.

Another way of calculating $\mu(s)$ is to travel along a given fiber, and to calculate the average fiber stress, which will be less than s since there are regions of the fiber near occasional breaks that are partially unloaded. Now, a segment of length $0 < x \leqslant s$ carries average fiber stress $x/2$ (as its load profile is a triangular shape), whereas a segment of length $x > s$ has a central portion of length $x - s$ where the stress is s, and end pieces totalling length s where the average stress is $s/2$. Thus, we obtain

$$\mu(s) = \int_0^s (x^2/2) p(s, x) \, dx + s \int_s^{\infty} \{(x - s) + s/2\} p(s, x) \, dx. \tag{12}$$

This result is quickly seen to be equivalent to (8) combined with (11) upon exchanging the order of integration.

Next, we turn to calculating the variance function $\Gamma_0(s)/n$ for a composite cross-section, were $\Gamma_0(s)$ is the variance in the stress of an arbitrarily selected fiber at that cross-section. (Note that we are altering slightly the notation in earlier papers (e.g. Phoenix and Raj, 1992).) This is the expected value of the square of the fiber stress at a cross-section minus the square of the mean fiber stress $\mu(s)$. Following the steps used in deriving (8) we get

$$\Gamma_0(s) = s^2 \int_{s/2}^{\infty} g(s, y) \, dy + \int_0^{s/2} (2y)^2 g(s, y) \, dy - \mu(s)^2. \tag{13}$$

On the other hand, we can travel along a given fiber, and calculate the mean square of the fiber stress (which will be less than s^2 since there are regions of the fiber near occasional breaks that are partially unloaded) and then subtract the square of the mean $\mu(s)^2$. In a segment of length $0 < x \leqslant s$ the average value of the square of the fiber stress is $x^3/3$ (since its load profile is actually a triangular shape), whereas a segment of length $x > s$ has a central portion of length $x - s$ where the stress is s^2, and end pieces totalling length s where the average square of the stress is $s^3/3$. Thus we obtain

$$\Gamma_0(s) = (1/3) \int_0^s x^3 p(s, x) \, dx + s^2 \int_s^{\infty} x p(s, x) \, dx$$

$$- (2/3) s^3 \int_s^{\infty} p(s, x) \, dx - \mu(s)^2. \tag{14}$$

Note that (13) combined with (11) is equivalent to (14) as can be seen by exchanging the order of integration.

There is one more function that we need to estimate, which is called the covariance function $\Gamma_\delta(s)/n$ for the stress at two composite cross-sections separated by a normalized distance $\delta > 0$ and where again the far field fiber stress is s. Note that δ will be scaled in terms of s because s is also the normalized exclusion zone length around a break. The covariance function $\Gamma_\delta(s)$ is the expected value of the product of the stresses at the two cross-sectional planes minus the square of the mean $\mu(s)^2$, and captures the key information on the correlation between the strengths of two nearby cross-sectional planes. Actually its local behavior for small δ is what is needed in determining the probability distribution for the weakest plane in the composite over length $L = m\delta_c$, using asymptotic theory for Gaussian processes given in Leadbetter *et al.* (1983).

Unfortunately, we cannot calculate this covariance function exactly because we do not have the joint distributions of the lengths of two or more adjacent fragments, and, in particular, we do not known if the lengths of adjacent fragments are statistically dependent or independent. Such a calculation would require greatly expanding the scope of the differential equations in Hui *et al.* (1995). On the other hand, on the basis of Monte Carlo simulations, Henstenburg and Phoenix (1989) saw negligible correlation in adjacent lengths. Thus we will take a simple approach along the lines used by Phoenix and Raj (1992) in developing their 'first' approximations subscripted by '1'. In essence we assume that the breaks approximately follow a Poisson process along the length with a certain rate $\vartheta(s)$ depending on s chosen specifically to give us the correct variance result, $\Gamma_0(s)$, for the special case $\delta = 0$. That is, the fragment lengths are assumed i.i.d. with an exponential distribution function with parameter $\vartheta(s)$. (The Poisson process assumption is actually valid for small s.) We never really have to calculate $\vartheta(s)$ since it is buried in $\Gamma_0(s)$, which we know. Our main result will turn out to be very simple.

Now the zone of interest spans from a distance $s/2$ to the left of the first plane to $s/2$ to the right of the second plane, so it has a total length of $\delta+s$. Under our Poisson process assumptions the number of breaks in this length follows a Poisson distribution, and given there are $j \geqslant 1$ breaks, these breaks are i.i.d. following a uniform distribution. Consider the case of no breaks occurring on this length. The probability of there being no breaks is $\exp\{-\vartheta(s)(\delta+s)\}$, and in the case the fiber carries load s at both planes so the product is s^2. Thus the contribution to $\Gamma_\delta(s)$ is

$$C_0 = s^2 \exp\{-\vartheta(s)(\delta+s)\}, \quad j = 0. \tag{15}$$

On the other hand, suppose that exactly one break has occurred over the fiber length $\delta+s$. The probability is $\vartheta(s)(\delta+s)\exp\{-\vartheta(s)(\delta+s)\}$, and we can take the break to be uniformly distributed over this length. Then this break can unload one, or the other, or even both of the planes depending on its longitudinal position. Now let z be its position relative to the midpoint of the length $\delta+s$, and assume $\delta < s/2$. Then the product of the fiber stress at each plane is

$$\begin{array}{ll}
-(\delta+2z)s & \text{for} \quad -(\delta+s)/2 < z < (\delta-s)/2, \\
-(\delta+2z)(\delta-2z) & \text{for} \quad (\delta-s)/2 < z < -\delta/2, \\
(\delta+2z)(\delta-2z) & \text{for} \quad -\delta/2 < z < \delta/2, \\
-(\delta+2z)(\delta-2z) & \text{for} \quad \delta/2 < z < (s-\delta)/2, \\
-s(\delta-2z) & \text{for} \quad (s-\delta)/2 < z < (\delta+s)/2.
\end{array}$$

Thus using this simple uniform distribution, the expected value of the product of the stress given one flaw is

$$\begin{aligned}
2s(s-\delta)\delta/(\delta+s) + 2(s^3/6 - s^2\delta/2 + 2\delta^3/3)/(\delta+s) + (2\delta^3/3)/(\delta+s) \\
= (s^2/3)\{1 + 2\delta/s - 8(\delta/s)^2 + O((\delta/s)^3)\}.
\end{aligned}$$

So, the contribution to $\Gamma_\delta(s)$ is

$$\begin{aligned}
C_1 = {} & (s^2/3)\{1 + 2\delta/s - 8(\delta/s)^2 \\
& + O((\delta/s)^3)\}\vartheta(s)(\delta+s)\exp\{-\vartheta(s)(\delta+s)\}, \quad j = 1.
\end{aligned} \tag{16}$$

In principle we could continue with $j = 2, 3, \ldots$ but the calculation becomes exceedingly complicated. For larger ρ, however, it turns out that the probability that $j \geqslant 2$ becomes negligible, and in any case more breaks means even lower stress products because of more unloading. So we can replace (16) by the overestimate

$$C'_1 = (s^2/3)\{1+2\delta/s-8(\delta/s)^2 + O((\delta/s)^3)\}[1-\exp\{-\vartheta(s)(\delta+s)\}], \quad j \geqslant 1. \tag{17}$$

Summing the two quantities in (15) and (17), the expected value of the product of the load at the two planes for small δ and large ρ is approximately

$$C_0 + C'_1 = s^2\exp\{-\vartheta(s)(\delta+s)\} + (s^2/3)\{1+2\delta/s - 8(\delta/s)^2 + O((\delta/s)^3)\}[1-\exp\{-\vartheta(s)(\delta+s)\}]. \tag{18}$$

Now we need to expand this further to isolate the effect of δ. To this end we note

$$\exp\{-\vartheta(s)(\delta+s)\} \approx \exp\{-\vartheta(s)s\}\{1-\vartheta(s)\,\delta\} \tag{19}$$

so that (18) becomes

$$\begin{aligned}C_0 + C'_1 \approx\ & s^2\exp\{-\vartheta(s)s\} + (s^2/3)[1-\exp\{-\vartheta(s)s\}] \\ & -(2s^2/3)\,\delta\vartheta(s)\exp\{-\vartheta(s)s\} \\ & +(2s/3)\,\delta[1-\exp\{-\vartheta(s)s\}] \\ & -(8\delta^2/3)[1-\exp\{-\vartheta(s)s\}].\end{aligned} \tag{20}$$

Now, to estimate the covariance $\Gamma_\delta(s)$ we must subtract the square of the mean fiber stress. First, we let

$$F(s) = 1-\exp\{-\vartheta(s)s\}, \quad s \geqslant 0. \tag{21}$$

Then, from Phoenix and Raj (1992), a good estimate of the mean is

$$\mu(s) \approx s\{1-F(s)/2\}. \tag{22}$$

(Numerically, (22) and (12) agree very closely for $\rho \geqslant 5$ as indicated in Hui *et al.* (1995).) We use this estimate because of its analytical simplicity in the present calculation. Thus we can write an estimate of $\Gamma_\delta(s)$ as (20) minus (22) squared, which is

$$\begin{aligned}\Gamma_\delta(s) \approx\ & s^2\{1-F(s)\} + (s^2/3)F(s) - s^2\{1-F(s)/2\}^2 \\ & -(2s^2/3)\,\delta\vartheta(s)\{1-F(s)\} + (2s/3)\,\delta F(s) - (8\delta^2/3)F(s) \\ =\ & s^2\{F(s)/3 - F(s)^2/4\} \\ & -(2s^2/3)\,\delta\vartheta(s)\{1-F(s)\} + (2s/3)\,\delta F(s) \\ & -(8\delta^2/3)F(s).\end{aligned} \tag{23}$$

From Phoenix and Raj (1992), the first term in (23) is a good estimate of the fiber variance function $\Gamma_0(s)$ given by (14), that is,

$$\Gamma_0(s) \approx s^2\{F(s)/3 - F(s)^2/4\} \tag{24}$$

as we see from later numerical comparisons. Thus we can write

$$\Gamma_\delta(s) \approx \Gamma_0(s)\{1-[(2s^2/3)\,\delta\vartheta(s)\{1-F(s)\} - (2s/3)\,\delta F(s)$$
$$-(8\delta^2/3)F(s)]\,[s^2\{F(s)/3 - F(s)^2/4\}]\}. \tag{25}$$

Finally we estimate the term in the large square brackets. First we use the approximation $F(s) \approx \vartheta(s)s$ noting that at composite failure this quantity is small for larger ρ. Upon expanding in $\vartheta(s)s$ we approximate (25) as

$$\Gamma_\delta(s) \approx \Gamma_0(s)\{1-8(\delta/s)^2 + o((\delta/s)^2)\} \tag{26}$$

where terms linear in δ have cancelled, and we have neglected terms in $\{\vartheta(s)s\}^2$. This estimate will suffice in later analysis.

The next quantity of interest is the maximum of the asymptotic mean stress $\mu(s)$ called μ^*, and is the composite stress where the load/strain curve begins to decrease in a composite with an infinite number of fibers n (where we recall that the strain is approximately s divided by the fiber Young's modulus). That is,

$$\mu^* = \max\{\mu(s): \quad s > 0\}. \tag{27}$$

The corresponding value of s is called s^*, so $\mu^* = \mu(s^*)$. As soon as s exceeds $s^*\sigma_c$, the composite will fail by fiber pullout at some collapse plane. No simple closed form expression for s^* is available, but carrying further an idea in Hui *et al.* (1995) we find that a very accurate approximation for all $\rho > 0$ turns out to be

$$s^* \approx \{(2/\rho)(4\rho+2)/(4\rho+1)\}^{1/(\rho-1)}. \tag{28}$$

Actually, this expression is asymptotically correct for both large ρ and for ρ near zero. Figure 1 plots this approximation against the exact solution numerically calculated in Hui *et al.* (1995), and the accuracy is seen to be exceptional.

As pointed out by Hui *et al.* (1995) there have been many attempts in the literature to estimate μ^*. All of the methods work well for $\rho > 10$, but even the best methods seriously diverge from the true value (usually overestimating μ^*) for $\rho < 2$. (See Fig. 5 in Hui *et al.* (1995).) These authors have obtained the three term asymptotic expansion

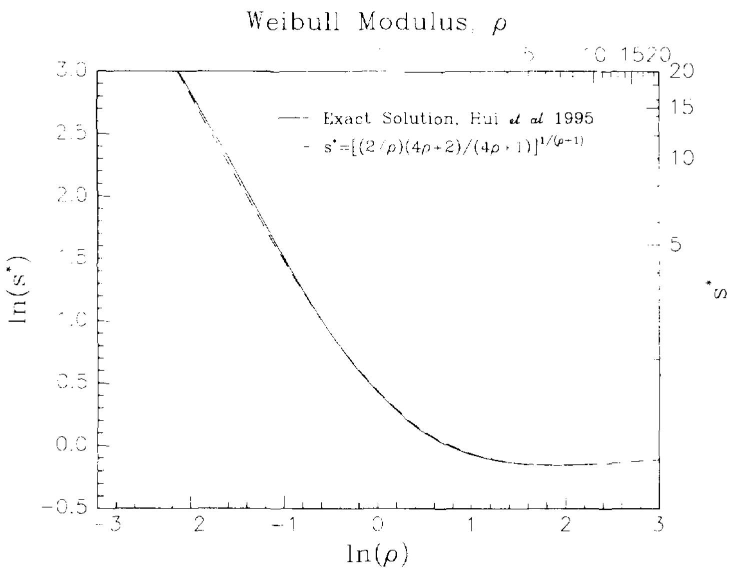

Fig. 1. Plot of fiber stress s^* (outside of the unloading/loading zone around a break) at composite collapse vs Weibull shape parameter ρ. Shown are the exact solution from Hui *et al.* (1995) and approximation (28).

$$\mu^* = \mu(s^*) \approx s^*[1+s^{*\rho+1}/2+\theta s^{*2\rho+2}]\exp\{-s^{*\rho+1}(1-\lambda s^{*\rho+1}/8)\} \tag{29a}$$

where

$$\lambda = \rho/(\rho+1) \tag{29b}$$

and

$$\theta = \{(7\rho-12)/(2\rho+3)\}/24. \tag{29c}$$

and this has less than 1% error for $\rho > 3$. Even at $\rho = 1$ the error is only about 8% low. On the other hand, we show in the Appendix that a Taylor series expansion for ρ near zero results in

$$\mu^*(\rho) \approx 1-0.922784335\rho+(1/2)1.534948649\rho^2+O(\rho^3) \tag{30}$$

being consistent with the analytical fact that $\mu^* = 1$ at $\rho = 0$. Figure 2 plots these approximations against the true μ^* obtained numerically by Hui *et al.* (1995). (Actually, they calculated no values for $0 < \rho < 0.5$.) Thus we have excellent approximations for μ^* say for $0 \leqslant \rho < 0.2$ and $\rho > 1.5$, respectively.

Next we evaluate the asymptotic, normalized, standard deviation for a composite cross-section. This is denoted γ_n^* and it is given by

$$\gamma_n^* = \{\Gamma_0(s^*)/n\}^{1/2}. \tag{31}$$

Thus we must evaluate $\Gamma_0(s)$ at $s = s^*$. This was done for $\rho \geqslant 0$ by numerically integrating in (14). Apart from $\mu^{*2} = \mu(s^*)^2$, which is already evaluated, there are three integrals to evaluate, namely

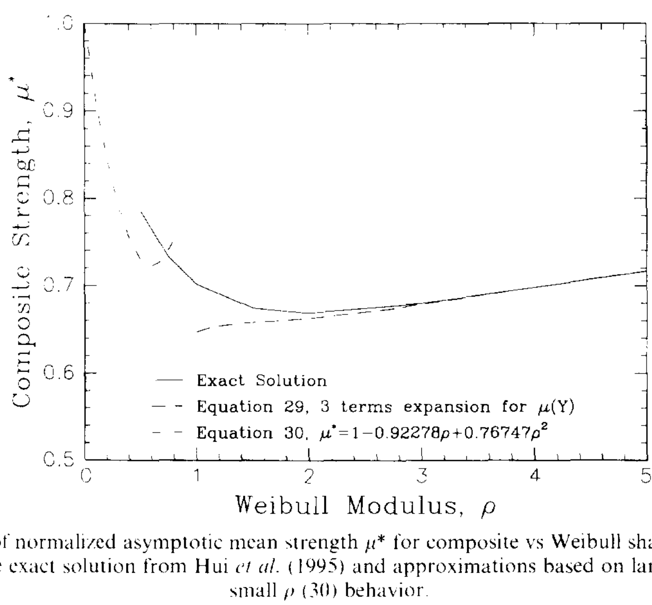

Fig. 2. Plot of normalized asymptotic mean strength μ^* for composite vs Weibull shape parameter ρ. Shown are exact solution from Hui *et al.* (1995) and approximations based on large ρ (29) and small ρ (30) behavior.

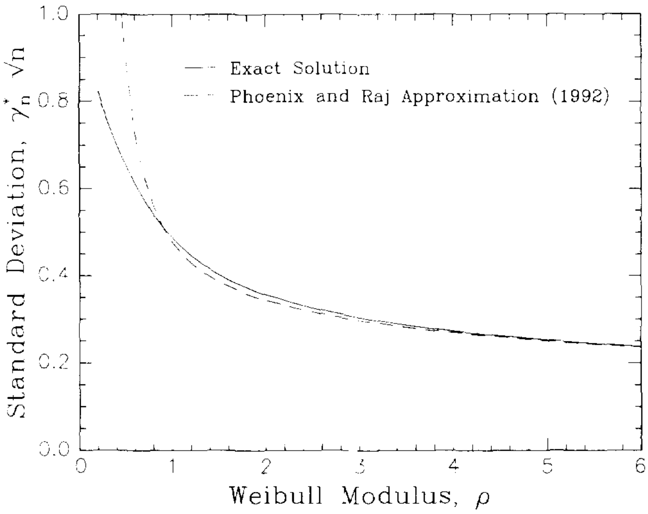

Fig. 3. Plot of normalized asymptotic standard deviation $n^{1/2}\gamma_n^*$ for composite strength vs Weibull shape parameter ρ. Shown are exact solution and an approximation based on Phoenix and Raj (1992).

$$I_1 = \int_0^{s} x^3 p(s, x)\,\mathrm{d}x, \tag{32a}$$

$$I_2 = \int_s^{t} xp(s, x)\,\mathrm{d}x, \tag{32b}$$

and

$$I_3 = \int_s^{t} p(s, x)\,\mathrm{d}x. \tag{32c}$$

These may be computed by brute force techniques, but they are computed much more efficiently using expansion techniques as described in Section 5.4 of Hui *et al.* (1995) for computing μ^*. Figure 3 shows a plot of $\gamma_n^* n^{1/2} = \{\Gamma_0(s^*)\}^{1/2}$. Also shown is an approximation for $\{\Gamma_0(s^*)\}^{1/2}$ based on one in Phoenix and Raj (1992) where $\Gamma_0(s)$ is given by (24) with $F(s)$ given by (22). From that work we take the simple approximation

$$\vartheta(s^*) \approx \vartheta_1(s^*) = s^{*\rho} \tag{33}$$

and we take s^* as given by (28). Figure 3 shows that this approximation works extremely well for $\rho > 0.8$.

Lastly we determine a refined mean μ_n^* and standard deviation γ_n^{**}, respectively, applicable when n is fairly small. The refinement for the mean, discussed in Phoenix and Raj (1995), is

$$\mu_n^* = \mu^* + \Delta_n^* \tag{34}$$

where

$$\Delta_n^* \approx n^{-2/3}\{(\rho/3)s^{*\rho+2} + [(1-\rho)/3]s^{*2\rho+3}\}^{2/3}\{(\rho+1)^2 s^{*\rho}[1-2s^{*\rho+1}]\}^{-1/3} \tag{35}$$

and s^* can be taken as (28). For $\rho > 4$, it is roughly $0.25n^{-2/3}$ and thus for $n = 50$ it amounts to about 2 or 3% of μ^* and for $n = 15$ it is about 5 or 6% of μ^*. Its effects will be noticeable in later numerical results.

A corrected standard deviation γ_n^{**} has not been derived for the present model under global load-sharing. However, we believe it is reasonable to adapt a result for classical, equal load-sharing bundles as given in McCartney and Smith (1983), and expect this result to give the proper scaling in n and to be close to the proper magnitude. Their revised standard deviation can be written as

$$\gamma_n^{**} = \gamma_n^*[1-(0.317)(\mu^*/\gamma_n^*)^2\rho^{-2/3}e^{4/(3\rho)}n^{-4/3}]^{1/2} \tag{36}$$

where, in their case, μ^* and γ_n^* are the mean and asymptotic standard deviation (large n) appropriate to equal load-sharing bundles. In the present context we take μ^* to be given by (27) or the approximations (29) or (30) as appropriate and γ_n^* to be given by (31) or subsequent approximation as appropriate.

Thus we have developed the key quantities in determining the strength distribution for a cross-section. The final assertion is that the strength at a cross-section is approximately normally distributed with mean μ_n^* and standard deviation γ_n^{**}, and, furthermore, that the covariance between two planes δ apart is given by $\Gamma_0(s^*)\{1-8(\delta/s^*)^2+o[(\delta/s^*)^2]\}/n$. Thus we have the main quantities enabling us in the next section to use certain results on minima in Gaussian processes, in order to estimate the distribution for strength of a long composite.

4. ASYMPTOTIC DISTRIBUTIONS FOR THE STRENGTH OF LONG COMPOSITES

We are now in a position to use some results given in chapter 8 of Leadbetter *et al.* (1983), particularly theorem 8.2.7 which allows us to construct an asymptotic approximation to $H_{m,n}(\sigma/\sigma_c)$, the distribution function for the strength of the composite. (See also the comment at the beginning of their chapter 11 to adapt results for the maximum to the minimum.) Our covariance function (24) has the structure assumed in the analysis there, with the correspondence being between their $r(\tau)$ and our $\Gamma_0(s^*)/n$ where their τ is our δ and where their $\lambda_2/2$ is our $8/s^{*2}$ in (26). Their theorem also assumes a mean of zero and a standard deviation of one, but we can change the location parameter by μ_n^* and rescale through multiplying by the asymptotic standard deviation γ_n^{**} to use their results. In particular we let

$$a_{m,n} = \gamma_n^{**}\,[2\log_e(m)]^{1/2} \tag{37a}$$

and

$$b_{m,n} = \mu_n^*-\gamma_n^{**}\{[2\log_e(m)]^{1/2}-\log_e(s^*\pi/2)/[2\log_e(m)]^{1/2}\}. \tag{37b}$$

(Note that our $a_{m,n}$ corresponds to their $1/a_T$ and our $b_{m,n}$ corresponds to their b_T.) Then the approximation to $H_{m,n}(\sigma/\sigma_c)$ is the double exponential form

$$H_{m,n}(\sigma/\sigma_c) \approx 1-\exp\{-\exp\{[(\sigma/\sigma_c)-b_{m,n}]/a_{m,n}\}\}, \quad \sigma \geqslant 0. \tag{37c}$$

As an estimate of the median, $\sigma_{m,n}^*$ (composite stress at probability of failure $H_{m,n}(\sigma/\sigma_c) = 1/2$) is

$$\sigma_{m,n}^*/\sigma_c \approx \mu_n^*-\gamma_n^{**}\{[2\log_e(m)]^{1/2}-\log_e(s^*\pi/2)/[2\log_e(m)]^{1/2}$$
$$-\log_e(\log_e(2))/[2\log_e(m)]^{1/2}\}. \tag{38}$$

The approximation (37c) has shortcomings in accuracy especially in the lower tail where it overestimates the probability of failure, so we now pursue a different approach that turns out to perform very well. (The above results will be referred to as the Gaussian process version.)

In the last section it was mentioned that the strength at a cross-sectional plane is approximately normally distributed with mean μ_n^* and standard deviation γ_n^{**}. Thus we consider the standardized stress variable

$$z = [(\sigma/\sigma_c) - \mu_n^*]/\gamma_n^{**}. \tag{39}$$

The standard normal density function (which has mean zero and standard deviation one) is

$$\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2), \quad -\infty < z < \infty \tag{40}$$

and its cumulative distribution function is

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)\,\mathrm{d}x, \quad -\infty < z < \infty. \tag{41}$$

For the lower tail of $\Phi(z)$, say for $z < -1$, Feller (1968) gives the result

$$\{1/|z| - 1/|z|^3\}\phi(z) \leqslant \Phi(z) \leqslant \phi(z)/|z|. \tag{42}$$

Thus we can approximate the lower tail of $\Phi(z)$ by

$$\Phi^*(z) = (2\pi)^{-1/2} \exp(-z^2/2)/|z|, \quad z \ll 0. \tag{43}$$

Recall that $H_{m,n}(\sigma/\sigma_c)$ is the distribution function for the strength of a composite of normalized length $m = L/\delta_c$. Letting $m' = m/\beta$ for some fixed constant $\beta > 0$, and using the approximation $(1-\Phi^*)^{m'} = \exp(-m'\Phi^*)$ for large m', we may construct the weakest-link approximation

$$H_{m,n}(\sigma/\sigma_c) \approx 1 - \exp\{-(m/\beta)\Phi^*([(\sigma/\sigma_c) - \mu_n^*]/\gamma_n^{**})\}, \quad \sigma \geqslant 0. \tag{44}$$

In essence we are assuming that the composite strength is given by the weakest of $m' = m/\beta$ cross-sections spaced $\beta\delta_c$ apart, where the parameter β is chosen so that the strengths at these cross-sections are effectively i.i.d. normal random variables, and there is a sufficient number of them to provide a good representation of the weakest-link effect. Now in order to study quantiles like the median chain strength (median strength of the composite), we consider a solution $\sigma_{m,\zeta}$ to

$$\Phi^*([(\sigma_{m,\zeta}/\sigma_c) - \mu_n^*]/\gamma_n^{**}) = \zeta/m' \tag{45}$$

where $\zeta > 0$ is a constant. An asymptotic result for $[(\sigma_{m,\zeta}/\sigma_c) - \mu_n^*]/\gamma_n^{**}$ (with $\mu_n^* = 0$, $\gamma_n^{**} = \sigma_c = 1$) is given in Cramér (1946), and yields (when rescaled to our case)

$$\begin{aligned} \sigma_{m,\zeta}/\sigma_c = \mu_n^* - \gamma_n^{**}\{&[2\log_e(m')]^{1/2} \\ &- \{\log_e(\log_e(m')) + \log_e(4\pi)\}/\{2[2\log_e(m')]^{1/2}\} \\ &- \log_e(\log_e(\zeta))/[2\log_e(m')]^{1/2} \\ &+ O(1/\log_e(m'))\}. \end{aligned} \tag{46}$$

For the median, $\sigma_{m,n}^*$ (composite stress at probability of failure of 1/2), ζ corresponds to $-\log_e(1/2) = \log_e(2)$, and using (46) we may write

$$\sigma^*_{m,n}/\sigma_\ell \approx \mu^*_n - \gamma^{**}_n\{[2\log_e(m/\beta)]^{1/2}$$
$$-\{\log_e(\log_e(m/\beta)) + \log_e(4\pi)\}/\{2[2\log_e(m/\beta)]^{1/2}\}$$
$$-\log_e(\log_e(2))/[2\log_e(m/\beta)]^{1/2}\}. \tag{47}$$

It is interesting to note that in the statistical theory of extremes for the minimum of $m' = m/\beta$, i.i.d. normal random variables with mean μ^*_n and standard deviation γ^{**}_n, (Leadbetter *et al.*, 1983) the right-hand side of (46) with $\log_e(\log_e(\zeta))$ replaced by $\log_e(\log_e(e)) = 0$, which we call $b_{m',n}$, ought to correspond to $b_{m,n}$ in (37b). The parameter $a_{m',n}$ is just $a_{m,n}$ with m' in place of m. Apart from the difference in m vs m' these quantities are not quite the same. Also for the median $\sigma^*_{m,n}$, the formulae (38) and (47) are not quite the same in structure (even after accounting for the additional parameter β in (47) through $m' = m/\beta$). We can, however, employ the following expansions for $m \gg \beta$ in (37b), namely

$$[2\log_e(m/\beta)]^{1/2} = [2\log_e(m) - 2\log_e(\beta)]^{1/2}$$
$$\approx [2\log_e(m)]^{1/2} - \log_e(\beta)/[2\log_e(m)]^{1/2}, \tag{48a}$$
$$\log_e\log_e(m/\beta) \approx \log_e\log_e(m) - \log_e(\beta)/\log_e(m) \tag{48b}$$

and

$$[2\log_e(m/\beta)]^{-1/2} = [2\log_e(m) - 2\log_e(\beta)]^{-1/2}$$
$$\approx [2\log_e(m)]^{-1/2}\{1 + \log_e(\beta)/[2\log_e(m)]\} \tag{48c}$$

to facilitate comparison. Substituting (48a) to (48c) into (47) and comparing like terms between (47) and (38) (upon ignoring terms of order $O(1/[\log_e(m)]^{3/2})$ or smaller) suggests that the quantity $-\log_e(s^*\pi/2)$ corresponds to $-\{\log_e\log_e(m) + \log_e(4\pi)\}/2 - \log_e(\beta)$ yielding

$$\beta \approx s^*\pi^{1/2}/[4\log_e(m)]$$
$$= 0.4431s^*/\log_e(m). \tag{49}$$

Thus, the difficulty that we see in comparing the two methods, a minimum in a Gaussian process vs a simple weakest link structure in terms of a chain of $m' = m/\beta$ bundles, is that there really is no obvious value for β that can be identified for use in the weakest link form. Of course in both formulae for the median, namely (38) and (47), the term $\gamma^{**}_n[2\log_e(m)]^{1/2}$ strongly dominates the asymptotics so in that sense the precise choice of β is of minor significance as the two medians will agree in magnitude as m grows very large, but the precise value is important for smaller $m' = m/\beta$. Later, we would like to rescale Monte Carlo simulation results for long composites, and this lack of simple correspondence creates small ambiguities. For example, (49) suggests $\beta \approx 0.4431s^*$ for $m = 3$ and $\beta \approx 0.13s^*$ for $m = 30$, if this method of comparison has any validity.

Pursuing this anomaly a little further, if we consider the quantity $\log_e(\log_e(m'))/\{2[2\log_e(m')]^{1/2}\}$, which is the term in (47) not appearing in (38), we calculate the values 0.133, 0.194, 0.224 and 0.244 for $m' = 5$, 10, 20 and 50, respectively, and these values are much smaller than those for $[2\log_e(m')]^{1/2}$. So, except for the smallest values of m (where the use of asymptotics would be questionable), this quantity is about 0.2. Even for $m = 1000$ it is only 0.26 and it remains so up to about $m = 100{,}000$ after which it slowly decays to zero. Thus, this ambiguity in pinning down an effective value of β in a comparison of the two methods amounts to little more than a small, fairly constant shift in the predicted median. We will consider this more in later numerical examples.

In any case, to compare these results to those of Monte Carlo simulation, we consider the 'reverse weakest-link transform'

$$\Phi_{m'}([(\sigma/\sigma_c)-\mu_n^*]/\gamma_n^{**}) = 1-\{1-H_{m,n}(\sigma/\sigma_c)\}^{1/m'} \tag{50}$$

where $H_{m,n}(\sigma/\sigma_c)$ is the empirical distribution function calculated from the data of the Monte Carlo simulation at given length $L = m\delta_c = (m'\beta)\delta_c$. The idea is to adjust β to superimpose $\Phi_{m'}([(\sigma/\sigma_c)-\mu_n^*]/\gamma_n^{**})$ onto a $\Phi([(\sigma/\sigma_c)-\mu_n^*]/\gamma_n^{**})$, the asymptotic normal distribution function for the strength of a cross-section, and achieve a good fit. In the next section we carry out various comparisons.

5. COMPARISON OF THEORY AND MONTE CARLO SIMULATIONS

In earlier work Ibnabdeljalil and Phoenix (1995) have developed a Monte Carlo simulation program for the failure of the composite. Its main features are summarized here as follows: the composite is assumed to have n fibers in parallel, and its length is $L = m\delta_c$, where $m > 0$ is an integer. It is partitioned into $n_{p,m}$ slabs of length δ_p such that

$$n_{p,m} = m\delta_c/\delta_p. \tag{51}$$

Also, $n_{p,1}$ is abbreviated as n_p, being the number of slabs of length δ_p in a characteristic length δ_c. Note that

$$\sigma_p = \sigma_c n_p^{1/\rho} = \sigma_0(l_0/\delta_p)^{1/\rho}, \tag{52}$$

which is the Weibull scale parameter for strength at length δ_p. In total there are $nn_{p,m}$ fiber elements in the composite. Generally, n_p must be chosen with some care, and to some extent a suitable value depends on the fiber shape parameter ρ and the length $L = n_{p,m}\delta_p$. In particular, as a composite specimen fails during a simulation run, the actual slip lengths should turn out to be about 10 times δ_p or more. This is because the stress on a fiber element will be taken as constant over its length, the stress will be evaluated at the element's center and the failure will occur there also.

Simulating the tensile failure of one composite 'specimen' of length L (one realization) begins with determining $nn_{p,m}$ realizations of the fiber strengths sampled independently from a Weibull distribution at length δ_p, that is, from the distribution function

$$F_p(\sigma) = 1-\exp\{-(\delta_p/l_0)(\sigma/\sigma_0)^\rho\}, \quad \sigma \geqslant 0, \tag{53}$$

and then normalizing each strength by σ_c. (This is equivalent to sampling from a Weibull distribution with shape parameter ρ and scale parameter $(n_p)^{1/\rho}$.) One realization of the failure of a composite specimen involves the following steps: (1) an applied stress equal to the strength of the weakest fiber element is applied uniformly to all $nn_{p,m}$ fiber elements. This causes the weakest element to fail. (2) The stress on that element is set to zero and its former stress is redistributed equally onto all fiber elements in the transverse plane (slab) of the break. (Generally this will be all elements that are not broken or slipping within the slip zone of another break along the same fiber.) Also, all fiber elements in the slip zone of this fiber break along the same fiber will experience a reduction in their stresses proportional to their center distances from the break divided by the slip length. These elements will see no further increases in stress throughout the duration of the test. Furthermore the lost stress of any element is distributed equally onto all fiber elements in the same transverse plane, so their stresses increase. (Generally these are again elements surviving and not slipping.) (3) The new stresses on all the fiber elements are then checked to see if any have been overloaded, that is, their strengths have been exceeded. If overloaded elements are found, then the most severely overloaded element is broken, and the stress is redistributed according to step (2). Then the new element stresses and strengths are compared again, and if any elements are found overloaded, the most severely overloaded (highest difference) is failed and the stress redistributed again. This process of failing overloaded fibers and redistributing stress is repeated until no overloaded fiber elements are found and the composite is in a stable state. (4) The stress on the composite is then increased producing proportional increases on every non-broken and non-slipping fiber element, and the increase is just enough to overload and break one element. This element is broken, and, if possible, the process repeats beginning with step (2). (5) Eventually, at some stage of iteration of the element breaking and stress redistribution process all the fiber elements in some transverse plane will be either broken or slipping. At that point, the composite can no longer take any additional load, and, thus, is considered failed. The tensile strength of the specimen is then taken as the stress level just prior to this discovery. Several hundred realizations of the strength of a composite of length $L = m\delta_c$ are obtained by this procedure. Note that mesh sensitivity tests are conducted for the various ρ values of interest in order to confirm an adequate mesh size whereby the results are insensitive to further refinement.

In summary, the Monte Carlo simulation program is free of certain assumptions made in the analysis with respect to (i) ignoring small undulations in the stress profiles of fibers outside exclusion zones, which actually occur for composites when the number of fibers n is small, and (ii) assuming a normal distribution for the strength of a composite cross-section.

We now compare some results from the simulation program to those from the theory. The comparison is first carried out for a typical Weibull shape parameter $\rho = 5$, for $n = 50$ fibers in the composite and for composite lengths of $m = L/\delta_c = 2, 5, 10, 20$, and 50. In total these simulation results took several hundred hours of CPU time on a Sun (Sparc 2) workstation. For $n = 50$, cases of m significantly larger than 50 are too time consuming to carry out.

Figure 4a–e shows reverse weakest-link scaling of the simulation results, using (50), for the cases $\beta\delta_c \approx 0.25\delta_c, 0.4\delta_c, 0.7\delta_c$, and δ_c. Here the idea is to estimate an effective link length $\beta\delta_c$. Also shown is the theoretical Gaussian (normal) approximation developed in Section 3. It is remarkable how well these empirical distributions actually superimpose for n ranging from 2 to 50. Given the discussion at the end of Section 4 regarding ambiguities in defining β in a weakest link view, this consistency might seem surprising. However it was pointed out there that the term causing ambiguity has an effect of about $0.2\gamma_n^{**} \approx 0.009$ for $n = 50$ and the range of m under consideration as compared to the mean $\mu_n^* = 0.74$. This would be barely noticeable being of the order of one division on these plots, which is actually the order of the variation in the simulation data. It would seem that $0.7\delta_c$ or $\beta = 0.7$ of Fig. 4c gives the best agreement to the theoretical Gaussian plot in terms of matching the overall slope and location. However, the lower tail is the most important region on these plots when considering the behavior of long composites and their reliabilities (lower tail behavior) through extrapolation. Later plots suggest that $\beta = 0.4$ is perhaps more appropriate for the lengths considered. It turns out that the corrections to the mean and standard deviation discussed at the end of Section 3 are important in these plots and noticeably improve the resolution. For $n = 50$ the finite size corrections to the mean (34) and standard deviation (36) are 2.5% and −16%, respectively.

Figure 5 shows the relationship between median lifetime and $m = m'\beta = L/\delta_c$. This figure amounts to a rotation of Fig. 4c clockwise by 90°. Also plotted are the theoretical medians (38) and (47) by the two methods. The Gaussian process version (38) appears to be the less accurate of the two in comparison to the simulation data, though it has no adjustable parameters. Agreement is good to values of m up to 10^4, and one has confidence that extrapolation would be valid to at least $m = 10^6$, which is far beyond the present capability of Monte Carlo simulation alone. Note that at $m = 100{,}000$, which would represent a long cable (if $\delta_c = 1$ mm, $L = 100$ m), the median strength is down by 18%.

Figure 6 shows a plot of the composite strength distribution, $H_{m,n}(\sigma/\sigma_c)$, for the case $n = 50$ and $m = 50$ on double exponential coordinates on which (37) plots as a straight line. Also shown is the data from Monte Carlo simulations as well as the weakest-link approximation (44) based on Gaussian (normal) lower tails for elements of length $\beta\delta_c$ where β is taken as 0.4. The double exponential approximation (37) is clearly conservative, especially in the lower tail as mentioned earlier. The weakest-link approximation (44) clearly performs much better.

Figure 7 shows a family of plots for the composite strength distribution $H_{m,n}(\sigma/\sigma_c)$ for $m = 5, 10, 20$, and 50. Monte Carlo data for each case is shown together with the

Gaussian/weakest-link approximation (44), again using $\beta = 0.4$. The agreement is excellent, though one can see the slight tendency to requiring a larger β value (say 0.7) for the case $m = 5$ near the median.

Figure 8a and b shows plots of the composite strength distribution, $H_{m,n}(\sigma/\sigma_c)$, for a much smaller number of fibers $n = 15$, and for the lengths $m = 5$ and 190, respectively. Again, double exponential coordinates are used on which (37) plots as a straight line. Also shown is the data from simulations as well as the Gaussian/weakest-link approximation (44) based on elements of length $\beta\delta_c$ where again β is taken as 0.4. The simulations on Fig. 8b required hundreds of hours on a Sun (Sparc 2) workstation. Again, the double exponential approximation (37) is clearly conservative especially in the lower tail, as mentioned earlier, and the weakest-link/Gaussian approximation (44) clearly performs much better. With a smaller number of fibers the drop in median strength corresponding to increasing m from 5 to 190 is substantial (from 0.70 to about 0.61). There is also some indication on Fig. 8b that an even smaller value of β would be appropriate as most of the simulation data lies to the left of the approximation (44). Also the data suggest that the assumption of a Gaussian lower tail of the link distribution is beginning to break down for this small value of n at larger m. This is understandable since, eventually, the deepest part of the lower tail is not

(a)

(b)

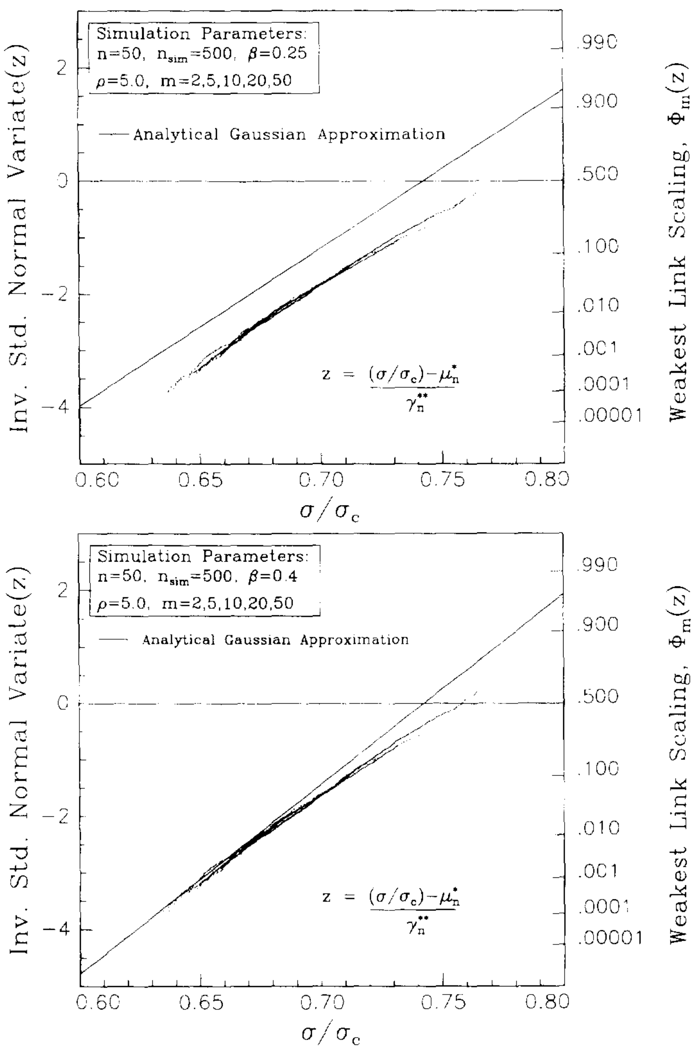

Fig. 4. Plot of 'reverse weakest link' transform (50) of normalized composite strength distribution obtained from Monte Carlo simulations on Gaussian coordinates for $n = 50$, $\rho = 5$ and $m = L/\delta_c = 2, 5, 10, 20, 50$. (a) $\beta = 0.25$, (b) $\beta = 0.4$, (c) $\beta = 0.7$, (d) $\beta = 1$. Also shown is the theoretically predicted asymptotic normal distribution for the strength of a composite cross-section with mean μ_n^* (34) and standard deviation γ_n^{**} (36). (*Continued overleaf.*)

(c)

(d)

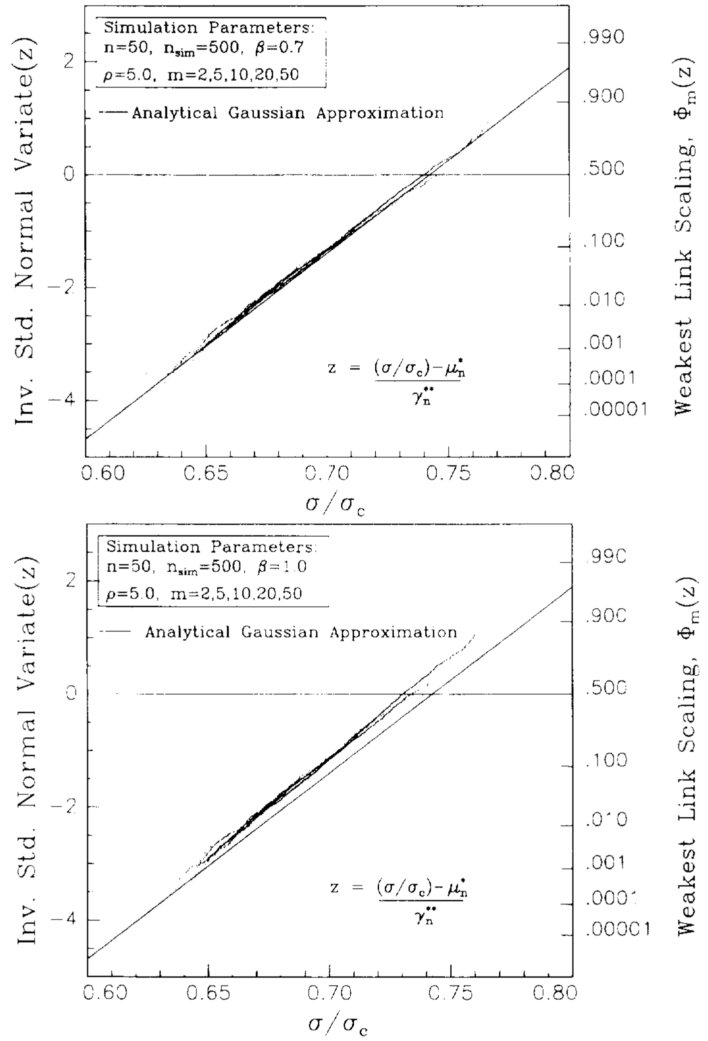

Fig. 4. (*Continued.*)

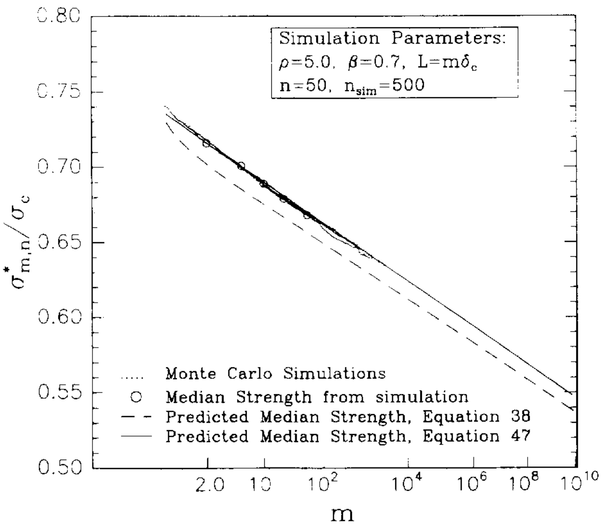

Fig. 5. Plot of median composite strength $\sigma^*_{m,n}/\sigma_c$ for two approximations (38) and (47) together with results from all the Monte Carlo simulations for $n = 50$, $\rho = 5$, $\beta = 0.7$ and $m = L/\delta_c = 2, 5, 10, 20, 50$.

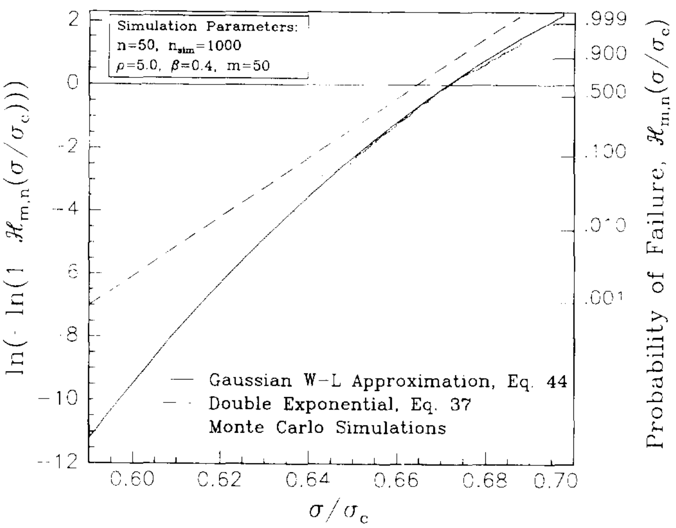

Fig. 6. Comparison of Gaussian weakest-link version (44) of $H_{m,n}(\sigma/\sigma_c)$ with double exponential version (37) on double exponential coordinates for $m = n = 50$, $\rho = 5$ and $\beta = 0.4$. Also shown are results from Monte Carlo simulations.

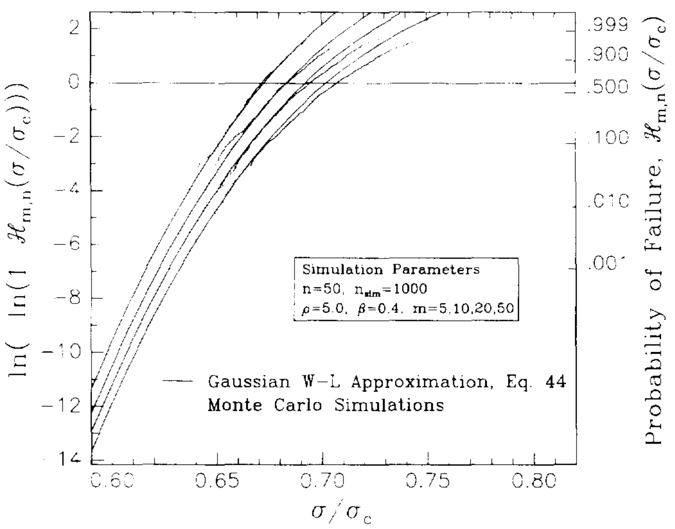

Fig. 7. Comparison of Gaussian weakest-link version (44) of $H_{m,n}(\sigma/\sigma_c)$ with Monte Carlo simulations on double exponential coordinates for $m = 5, 10, 20, 50$, $n = 50$, $\rho = 5$ and $\beta = 0.4$.

really Gaussian at all, since strengths cannot be negative. In any case, deeper into the lower tail, the approximations will tend to be more conservative.

6. CONCLUSIONS

We have developed an analysis for the probability distribution for the strength of a relatively long, brittle fiber brittle matrix composite. The analysis came in two versions: one was based on a weakest link model in terms of short, statistically independent composite elements, each of which had an approximately Gaussian (normal) distribution for strength; the other was based on weakest cross-section analysis in terms of a Gaussian process formulation for cross-section strength vs longitudinal position along the composite. These two methods gave comparable results, though the Gaussian process version with no adjustable parameter β tended to be less accurate and more conservative. The analytical predictions embodied in the most accurate form (45), and the simpler but less accurate form (37), were compared favorably to numerical results from Monte Carlo simulation. Also, quite simple formulas resulted for the size effect in the median strength both in terms of

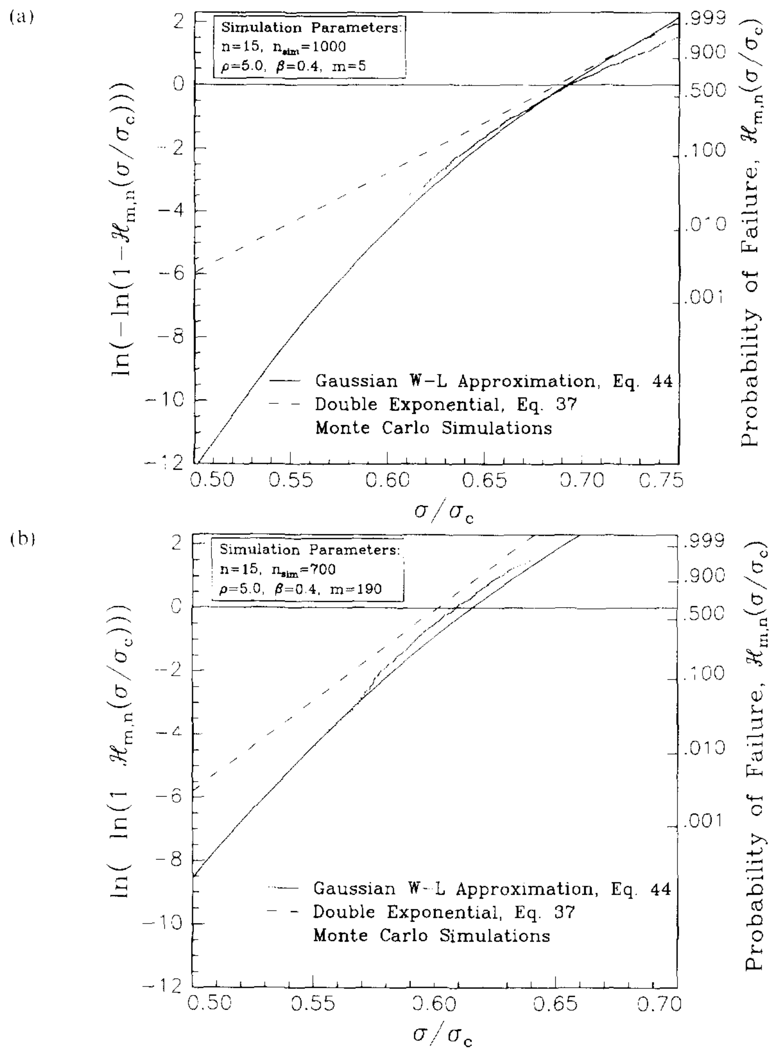

Fig. 8. Comparison of Gaussian weakest-link version (44) of $H_{m,n}(\sigma/\sigma_c)$ with double exponential version (37) and Monte Carlo simulation results on double exponential coordinates for $n = 15$, $\rho = 5$ and $\beta = 0.4$: (a) $m = 5$; (b) $m = 190$.

composite length and number of fibers in a cross-section. To achieve high accuracy it was necessary to have extremely accurate results for the asymptotic mean and standard deviation of the strength at a composite cross-section.

Values of $\beta\delta_c$ from $0.7\delta_c$ to $0.4\delta_c$ seems reasonable in view of the fact that the effective exclusion zone around a break is about $s^*\delta_c = 0.88\delta_c$ so the unloading length is half this at $0.44\delta_c$. On the other hand, in the Gaussian process version, as $m \to \infty$ only the local decay of $\Gamma_\delta(s)$ near $\delta = 0$ is used, but the quadratic approximation (26) substantially overestimates this decay (underestimates the true covariance) for larger δ, say $\delta = 0.5\delta_c$, and this may be the cause of disagreement for moderate values of m where the Gaussian process version seems conservative. At the same time, comparison between the two versions against Monte Carlo simulations suggested that β cannot really be viewed as a fixed parameter, but rather a parameter whose value decreases slowly with increasing m, meaning that the effective number of embedded links per unit length actually grows slowly. This is also a conclusion suggested by (49) which suggests that β must eventually decrease as $1/\log_e m$. Of course (49) can be seen to under-predict the effective values of β found here, but it is based on an asymptotic analysis for large m many orders of magnitude beyond what can be simulated. In any case agreement between theory and simulation is excellent lending confidence to extrapolations of median strength and high-reliability (lower tails of the distribution) to larger composites.

If the calculations were repeated for smaller ρ, say $\rho < 3$, then we would have to use μ^* and γ_n^* in place of μ_n^* and γ_n^{**}, that is, abandon the corrections for small n since the approximations for these will be inaccurate for small ρ. This would reduce the accuracy for smaller n, but fortunately in the direction of making the results more conservative.

Lastly we point out that in a time dependent version of the problem, Ibnabdeljalil and Phoenix (1995) concluded that under similar circumstances the effective link length was about $0.2\,\delta_c$, which is half the value arrived at here. They, however, had no estimates of the asymptotic mean and standard deviation, and the estimate $0.2\,\delta_c$ was based on getting the straightest plot on lognormal coordinates in the reverse weakest-link transform. However, differences between $\beta = 0.2$ and $\beta = 0.4$ are minor.

Acknowledgements C.-Y. Hui, and S. L. Phoenix acknowledge the support of the U.S. Air Force Office of Scientific Research, University Research Initiative (Grant no. F49620-93-1-0235). M. Ibnabdeljalil acknowledges the support of the U.S. Air Force Office of Scientific Research (Grant no. F49620-95-1-0158).

REFERENCES

Cramér, H. (1946). *Mathematical Methods of Statistics*, pp. 374–377. Princeton University Press, Princeton, NJ.

Curtin, W. A. (1991). Theory of mechanical properties of ceramic-matrix composites. *J. Am. Ceramic Soc.* **74**, 2837–2845.

Curtin, W. A. (1993). The 'tough' to brittle transition in brittle matrix composites. *J. Mech. Phys. Solids* **41**, 217–245.

Daniels, H. E. (1945). The statistical theory of the strength of bundles of threads. *Proc. R. Soc. A* **183**, 405–435.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, 3rd edition, pp. 174–179. John Wiley & Sons, New York.

Gücer, D. E. and Gurland, J. (1962). Comparison of the statistics of two fracture modes. *J. Mech. Phys. Solids* **10**, 365–373.

Henstenburg, R. B. and Phoenix, S. L. (1989). Interfacial shear strength studies using the single filament composite test. Part II: a probability model and Monte Carlo simulation. *Polym. Comp.* **10**, 389–408.

Hui, C.-Y., Phoenix, S. L., Ibnabdeljalil, M. and Smith, R. L. (1995). An exact closed form solution for fragmentation of Weibull fibers in a single filament composite with applications to fiber reinforced ceramics. *J. Mech. Phys. Solids* **43**, 1551–1585.

Ibnabdeljalil, M. and Phoenix, S. L. (1995a). Scalings in the statistical failure of brittle matrix composites with discontinuous fibers. *Acta Metall. Mat.* **43**, 2975–2983.

Ibnabdeljalil, M. and Phoenix, S. L. (1995b). Creep rupture of brittle matrix composites reinforced with time dependent fibers: scalings and Monte Carlo simulations. *J. Mech. Phys. Solids* **43**, 897–931.

Leadbetter, M. R., Lindgren, G. and Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences*, pp. 171–172, 205–206. Springer-Verlag, New York.

McCartney, L. N. and Smith, R. L. (1983). Statistical theory of the strength of fiber bundles. *J. Appl. Mech.* **105**, 601–630.

Phoenix, S. L. (1993). Statistical issues in the fracture of brittle-matrix fibrous composites. *Comp. Sci. Technol.* **48**, 65–80.

Phoenix, S. L. and Raj, R. (1992). Scalings in fracture probabilities for a brittle matrix fiber composite. *Acta Metall. Mat.* **40**, 2813–2828.

Smith, R. L. and Phoenix, S. L. (1981). Asymptotic distributions for the failure of fibrous materials under series-parallel structure and equal load-sharing. *J. Appl. Mech.* **103**, 75–82.

APPENDIX

We now justify the approximation of $\mu^*(\rho)$ for small ρ given by (30). Only some of the more difficult steps will be given in detail. The Taylor series expansion of $\mu^*(\rho)$ about $\rho = 0$ can be obtained by computing the nth derivative of

$$\mu^*(\rho) = A(\rho)\int_0^{[illegible]} Y^{[illegible]} \exp\left[\frac{2Y}{\rho - 1}\right]\Psi(Y,\rho)\rho R(Y,\rho)\,\mathrm{d}Y \tag{A1a}$$

with respect to ρ and evaluating it at $\rho = 0$, where

$$A(\rho) = \frac{2^{[illegible]}}{1+\rho}, \tag{A1b}$$

$$\rho R(Y,\rho) = (1-\rho)\ \ \rho Y\ 2 + \rho(1 - e^{[illegible]})\ Y. \tag{A1c}$$

$$\Psi(Y,\rho) = \exp\left[-\frac{2}{1+\rho}\int_0^Y \frac{1-e^{-t}}{t}\,dt\right], \tag{A1d}$$

and

$$Y^*(\rho) = (s^*)^{\rho+1}/2. \tag{A1e}$$

It should be noted that $s^*(\rho) \to \infty$ as $\rho \to 0$ so that $Y^*(\rho) \to \infty$ as $\rho \to 0$. Also, the integrand in (A1a) vanishes exponentially fast for all $\rho \geqslant 0$ as $Y \to \infty$. After some straightforward but tedious calculations we find

$$\left.\frac{d\mu^*}{d\rho}\right|_{\rho=0} = -\int_0^\infty e^{-t}\ln t\,dt - \frac{3}{2}. \tag{A2}$$

The integral is the well known Euler–Mascheroni constant $\gamma = 0.5772156649\ldots$. The evaluation of $(d^2\mu^*/d\rho^2)|_{\rho=0}$ is carried out in the same way, except that the calculation becomes extremely messy. The result is

$$\left.\frac{d^2\mu^*}{d\rho^2}\right|_{\rho=0} = \sum_{i=1}^{5} L_i, \tag{A3a}$$

where

$$L_1 = (\tfrac{5}{2} - 2\gamma)(1+\ln 2) - (\ln 2)^2, \tag{A3b}$$

$$L_2 = 4\left[-\frac{5}{24} - \frac{5\gamma}{8} - \frac{5\ln 2}{8} + \frac{5}{4}\ln\frac{3}{2} - \frac{(\ln 2)^2}{2} + \gamma\ln\frac{3}{2} + \frac{(\ln 3)^2}{2} - L_3\right], \tag{A3c}$$

$$L_3 = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[\frac{1}{2^n} - \frac{1}{3^n}\right], \tag{A3d}$$

$$L_4 = 2\left\{\begin{array}{l} 1 + \dfrac{1}{2}\left\{(\ln 2)^2 + 2\gamma\ln 2 + \gamma^2 + \dfrac{\pi^2}{6}\right\} - 4L_3 - (-\gamma - \ln 2 + 1) \\ -\left(2\ln\dfrac{3}{2} + \dfrac{2}{3}\right) - 2\left(\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2 2^k} - \gamma\ln\frac{3}{2} + \frac{(\ln 2)^2}{2} - \frac{(\ln 3)^2}{2}\right)\end{array}\right\}, \tag{A3e}$$

and

$$L_5 = 2(-1-\gamma + 2\ln 3 - 3\ln 2). \tag{A3f}$$

Note that $(d^2\mu^*/d\rho^2)|_{\rho=0}$ is independent of L_3 due to a cancellation of terms when L_2 is added to L_4. In carrying out that calculations to yield (A3), there are various integrals that must be evaluated that are not available in the literature. These integrals are recorded below:

$$\int_0^\infty e^{-ax}\left[\int_0^x\left(\frac{1-e^{-t}}{t}\right)dt\right]dx = \frac{1}{a}\ln\left(\frac{a+1}{a}\right), \tag{A4a}$$

$$\int_0^\infty e^{-ax}\left(\frac{1-e^{-x}}{x}\right)dx = \ln\left(\frac{a+1}{a}\right), \tag{A4b}$$

$$\int_0^\infty xe^{-ax}\left[\int_0^x\left(\frac{1-e^{-t}}{t}\right)dt\right]dx = \frac{1}{a^2}\left[\ln\left(\frac{a+1}{a}\right) + \frac{1}{a+1}\right], \tag{A4c}$$

$$\int_0^\infty (\ln x)e^{-ax}\left(\frac{1-e^{-x}}{x}\right)dx = -\gamma\ln\left(\frac{a+1}{a}\right) + \frac{(\ln a)^2}{2} - \frac{[\ln(a+1)]^2}{2}, \tag{A4d}$$

$$\int_0^\infty e^{-ax}\ln x\,dx = \frac{-[\gamma + \ln a]}{a}, \tag{A4e}$$

$$\int_0^\infty e^{-ax}x\ln x\,dx = \frac{1}{a^2}[1-\gamma-\ln a], \tag{A4f}$$

$$\int_0^\infty e^{-ax}(\ln x)^2\,dx = \frac{1}{a}\left[(\ln a)^2 + 2\gamma\ln a + \gamma^2 + \frac{\pi^2}{6}\right]. \tag{A4g}$$

$$\int_0^\infty \frac{e^{-ay}}{y}\left[\int_0^y \left(\frac{1-e^{-t}}{t}\right)dt\right]dy = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2 a^k}, \tag{A4h}$$

and

$$\int_0^\infty e^{-ay}\left(\frac{1-e^{-y}}{y}\right)\left[\int_0^y \left(\frac{1-e^{-t}}{t}\right)dt\right]dy = \frac{a}{2}\int_0^\infty e^{-ay}\left[\int_0^y \left(\frac{1-e^{-t}}{t}\right)dt\right]^2 dy, \tag{A4i}$$

where $a > 0$ in all the expression above. It should be noted that $a = 2$ in the calculations leading to (A3). These integrals are evaluated using combinations of integration by parts, change of variables and differentiation of the integrals with respect to a parameter (i.e. a). With the exception of the integral

$$\int_0^\infty \frac{e^{-ay}}{y}\left[\int_0^y \left(\frac{1-e^{-t}}{t}\right)dt\right]dy,$$

all the other integrals can be express in terms of the Euler–Mascheroni constant γ and elementary functions. The series representation of the integral

$$\int_0^\infty \frac{e^{-ay}}{y}\left[\int_0^y \left(\frac{1-e^{-t}}{t}\right)dt\right]dy,$$

i.e.

$$\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2 a^k}$$

converges extremely rapidly for $a = 2$, and can be easily computed to any degree of accuracy.

To save space, only two of the more difficult integral identities are shown below. We first verify (A4g). Let $I(\alpha) \equiv \int_0^\infty e^{-ay} y^\alpha\, dy$, and note that $(d^2 I/d\alpha^2)|_{\alpha=0} = \int_0^\infty e^{-ay}(\ln y)^2\, dy$. Furthermore, a change in variable leads to

$$I(\alpha) = a^{-\alpha-1}\int_0^\infty e^{-u}u^\alpha\, du = a^{-\alpha-1}\Gamma(\alpha+1).$$

Differentiating this expression twice with respect to α and evaluating at $\alpha = 0$ yields

$$\left.\frac{d^2 I}{d\alpha^2}\right|_{\alpha=0} = \frac{1}{a}[(\ln a)^2 - 2\Gamma'(1)\ln a + \Gamma''(1)].$$

Using the well known identities $\Gamma'(1) = -\gamma$ and $\Gamma''(1) = \gamma^2 + (\pi^2/6)$ completes the proof of (A4g).

Next, we verify the integral (A4d). We let

$$J(a) \equiv \int_0^\infty (\ln y)\, e^{-ay}\left(\frac{1-e^{-y}}{y}\right)dy,$$

then we find that

$$\frac{dJ}{da} = -\int_0^\infty (\ln y)\, e^{-ay}(1-e^{-y})\, dy.$$

Using

$$\gamma = -\int_0^\infty e^{-t}\ln t\, dt$$

we find that

$$\frac{dJ}{da} = \frac{\gamma}{a} + \frac{\ln a}{a} - \frac{\gamma}{a+1} - \frac{\ln(a+1)}{a+1}.$$

Solution of this differential equation gives

$$J(a) = \int_0^\infty (\ln y)\,\mathrm{e}^{-ay}\left(\frac{1-\mathrm{e}^{-y}}{y}\right)\mathrm{d}y = -\gamma\ln\left(\frac{a+1}{a}\right)+\frac{(\ln a)^2}{2}-\frac{[\ln(a+1)]^2}{2}.$$

The constant of integration is set to zero since $J(a=\infty)=0$, thus concluding the proof of (A4d).

Lastly, the first three terms of the Taylor series expansion of $\mu^*(\rho)$ are given by

$$\mu^*(\rho) = \mu^*(0)+\left.\frac{\mathrm{d}\mu^*}{\mathrm{d}\rho}\right|_{\rho=0}\rho+\frac{1}{2}\left.\frac{\mathrm{d}^2\mu^*}{\mathrm{d}\rho^2}\right|_{\rho=0}\rho^2. \tag{A5}$$

Using (A2) and (A3) the final result becomes

$$\mu^*(\rho) = 1-0.922784335\rho+\frac{1.534948649}{2}\rho^2+O(\rho^3). \tag{A6}$$